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Information scan of quantum states based on entropy-power uncertainty relations

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Abstract. We use Rényi-entropy-power-based uncertainty relations to show how the information probability distribution associated with a quantum state can be reconstructed in a process that is analogous to quantum-state tomography. We illustrate our point with the so-called “cat states”, which are of both fundamental interest and practical use in schemes such as quantum metrology, but are not well described by standard variance-based approaches.

1. Introduction

There has been a recent upsurge of interest in quantum-mechanical (QM) uncertainty relations (URs) catalyzed by new ideas from (quantum) information theory [1, 2, 3, 4, 5], functional analysis [6, 7] and cosmology [8, 9, 10] as well as experiments that have observed violations of Heisenberg’s error-disturbance uncertainty relations [11, 12, 13, 14, 15]. Historically the most popular quantifier of quantum uncertainty has been *variance* because of its simplicity and ubiquity in probability theory. The variance determines the measure of uncertainty in terms of the fluctuation (or spread) around the mean value which, while useful for many distributions, does not provide a sensible measure of uncertainty in a number of important situations including multimodal [1, 2, 3] and heavy-tailed distributions [2, 3, 4]. To deal with this, a multitude of alternative measures of uncertainty have emerged in QM. Among these, a particularly prominent role is played by information entropies such as the Shannon entropy [16], Rényi entropy [3, 16], Tsallis entropy [17], associated differential entropies and their quantum-information generalizations [2, 3, 5].

In Ref. [2] we introduced an infinite tower of mutually distinct (generally irreducible) Rényi entropy-power-based URs (REPURs). The conventional URs based on variances (Robertson–Schrödinger URs [18, 19]) and Shannon differential entropies (Hirschman and Białynicki-Birula URs [5, 20]) naturally appeared as special cases in this hierarchy of REPURs. The concept of *entropy power* (EP) was introduced in Shannon’s seminal 1948 paper [21] in order to formulate information theory for continuous random variables. Since then, the EP has proved to be essential in a number of classical and quantum information-theoretic applications ranging from interference channels to secrecy capacity [22, 23, 24, 25, 26]. Apart from its role in information theory, the EP has found wide use in pure mathematics, namely in the theory of inequalities and mathematical statistics and estimation theory [27].



In Refs. [2, 3] we addressed the following two questions: Assuming one is able to control Rényi EPs of various orders; i) how does this set of EPs constrain the underlying state distribution and ii) how do the ensuing REPURs restrict the state distributions of conjugate variables? To answer these questions we invoked the concept of the *information distribution* associated with a given quantum state. The latter contains a complete “information scan” of the underlying state distribution. Here we concentrate on the point i) by extending the analysis of [2, 3]. In particular, we show how the information distribution associated with a given quantum state can be numerically reconstructed from EPs in a process that is akin to a quantum-state tomography.

The paper is structured as follows. We begin in Sec. 2 by introducing the concept of Rényi’s EP, which is done in the context of estimation theory by generalizing the notion of Fisher information (FI) using a Rényi entropy version of De Bruijn’s identity. In this regard, a particularly notable role is played by the so-called *escort distribution*, which appears naturally in the definition of higher-order *score functions* and the ensuing generalized Cramér–Rao’s inequalities. After this, in Sec. 3 we introduce the concepts of information distribution and in Sec. 4 we show how cumulants of the information distribution can be obtained from knowledge of the EPs. With the cumulants at hand, one can reconstruct the underlying information distribution in a process which we call an *information scan*. Details of how this can explicitly be realized for quantum state PDFs is provided in Sec. 5. This is done with the help of a generalized Gram–Charlier A and the Edgeworth expansion. In Sec. 6 we illustrate the inner workings of the information scan using the example of a so-called *cat state*. A cat state is a superposition of the vacuum state and a coherent state of the electromagnetic field; two cases are studied comprising of different probabilistic weightings of the superposition state corresponding to *balanced* and *unbalanced* cat states. A short conclusion is presented in Sec. 7.

2. Cramér–Rao inequality and Rényi entropy powers

In Ref. [2, 3] we derived the Rényi-entropy-power-based QM uncertainty relations using the Babenko–Beckner inequality. In this section we provide another version of the proof which will hinge on the concept of FI and the (generalized) Cramér–Rao inequality. The derivation we outline fits the mathematical framework used in quantum metrology. Firstly, we recall that the Fisher information matrix $\mathbb{J}(\mathcal{X})$ of a random vector $\{\mathcal{X}_i\}$ in \mathbb{R}^D with the PDF $\mathcal{F}(\mathbf{x})$ is

$$\mathbb{J}(\mathcal{X}) = \text{cov}(\mathbf{V}(\mathcal{X})), \quad (1)$$

where the covariance matrix is associated with the random zero-mean vector (*score vector*),

$$\mathbf{V}(\mathbf{x}) = \nabla \mathcal{F}(\mathbf{x}) / \mathcal{F}(\mathbf{x}). \quad (2)$$

The ensuing FI $J(\mathcal{X})$ is a trace of $\mathbb{J}(\mathcal{X})$, i.e.

$$J(\mathcal{X}) = \text{Tr}(\mathbb{J}(\mathcal{X})) = \text{var}(\mathbf{V}(\mathcal{X})) = \mathbb{E}(\mathbf{V}^2(\mathcal{X})). \quad (3)$$

Both the FI and FI matrix can be conveniently related to Shannon’s differential entropy via De Bruin’s identity [28].

De Bruin’s identity: Let $\{\mathcal{X}_i\}$ be a random vector in \mathbb{R}^D with the PDF $\mathcal{F}(\mathbf{x})$ and let $\{\mathcal{Z}_i^G\}$ be a Gaussian noise vector with zero mean and unit-covariance matrix, independent of $\{\mathcal{X}_i\}$. Then

$$\frac{d}{d\epsilon} \mathcal{H}(\mathcal{X} + \sqrt{\epsilon} \mathcal{Z}^G) |_{\epsilon=0} = \frac{1}{2} J(\mathcal{X}), \quad (4)$$

where $\mathcal{H} = - \int_{\mathbb{R}^D} \mathcal{F}(\mathbf{x}) \log \mathcal{F}(\mathbf{x}) d\mathbf{x}$ is Shannon’s differential entropy (measured in *nats*). In the case when the independent additive noise $\{\mathcal{Z}_i\}$ is non-Gaussian with zero mean and covariance matrix $\mathbf{\Sigma} = \text{cov}(\mathcal{Z})$ then the following generalization holds

$$\frac{d}{d\epsilon} \mathcal{H}(\mathcal{X} + \sqrt{\epsilon} \mathcal{Z}) |_{\epsilon=0} = \frac{1}{2} \text{Tr}(\mathbb{J}(\mathcal{X}) \mathbf{\Sigma}). \quad (5)$$

The crux of De Bruin's identity is that it provides a very useful intuitive interpretation of the Fisher information, namely that it quantifies the sensitivity of transmitted (Shannon) information to an arbitrary independent additive noise. An important aspect to note is that this quantifier of sensitivity depends only on the covariance of the noise vector and so it is independent of the shape of the noise distribution. This is because De Bruin's identity remains unchanged for both Gaussian and non-Gaussian additive noise with the same covariance matrix. We now prove that the shape of the noise distribution can be further quantified by the Rényi-entropy-based FI matrix. Indeed, the following statement holds:

Generalized De Bruin's identity: Let $\{\mathcal{X}_i\}$ be a random vector in \mathbb{R}^D with the PDF $\mathcal{F}(\mathbf{x})$ and let $\{\mathcal{Z}_i\}$ be an independent (generally non-Gaussian) noise vector with the zero mean and covariance matrix $\mathbf{\Sigma} = \text{cov}(\mathcal{Z})$, then

$$\frac{d}{d\epsilon} \mathcal{I}_q(\mathcal{X} + \sqrt{\epsilon} \mathcal{Z})|_{\epsilon=0} = \frac{1}{2q} \text{Tr}(\mathbb{J}_q(\mathcal{X})\mathbf{\Sigma}), \quad (6)$$

where $\mathcal{I}_q = \frac{1}{1-q} \log \int_{\mathbb{R}^D} \mathcal{F}^q(\mathbf{x}) d\mathbf{x}$ is Rényi's differential entropy (measured in *nats*) with $\mathcal{I}_1 = \mathcal{H}$. The generalized FI matrix of order q has the explicit form

$$\mathbb{J}_q(\mathcal{X}) = \text{cov}_q(\mathbf{V}_q(\mathcal{X})), \quad (7)$$

with the score vector

$$\mathbf{V}_q(\mathbf{x}) = \nabla \rho_q(\mathbf{x}) / \rho_q(\mathbf{x}) = q \nabla \mathcal{F}(\mathbf{x}) / \mathcal{F}(\mathbf{x}) = q \mathbf{V}(\mathbf{x}). \quad (8)$$

Here $\rho_q = \mathcal{F}^q / \int_{\mathbb{R}^D} \mathcal{F}^q d\mathbf{x}$ is the so-called escort distribution [29]. The “ cov_q ” denotes the covariance matrix computed with respect to ρ_q . A proof of this generalized De Bruin's identity is provided in [30]. Finally, as in the Shannon case we define the FI of order q — denoted as $J_q(\mathcal{X})$, via De Bruin's identity with a unit noise covariance matrix. This gives

$$\text{Tr}(\mathbb{J}_q(\mathcal{X})) \equiv J_q(\mathcal{X}). \quad (9)$$

As for Shannon's information theory, there also exists a close connection between the FI matrix $\mathbb{J}_q(\mathcal{X})$ and the corresponding Rényi entropy power $N_p(\mathcal{X})$. Rényi's entropy power is defined as the solution of the equation

$$\mathcal{I}_p(\mathcal{X}) = \mathcal{I}_p\left(\sqrt{N_p(\mathcal{X})} \cdot \mathcal{Z}^G\right), \quad (10)$$

where $\{\mathcal{Z}_i^G\}$ represents a Gaussian random vector with zero mean and unit covariance matrix. So, $N_p(\mathcal{X})$ denotes the variance of a would be Gaussian distribution that has the same Rényi information content as the random vector $\{\mathcal{X}_i\}$ described by the PDF $\mathcal{F}(\mathbf{x})$. Expression (10) was studied in [2, 3, 31] where it was shown that the only class of solutions of (10) is

$$N_p(\mathcal{X}) = \frac{1}{2\pi} p^{-p'/p} \exp\left(\frac{2}{D} \mathcal{I}_p(\mathcal{X})\right), \quad (11)$$

with $1/p + 1/p' = 1$ and $p \in \mathbb{R}^+$. In addition, when $p \rightarrow 1_+$ one has $N_p(\mathcal{X}) \rightarrow N(\mathcal{X})$, where $N(\mathcal{X})$ is the conventional Shannon entropy power [32].

Generalized isoperimetric inequality: Let $\{\mathcal{X}_i\}$ be a random vector in \mathbb{R}^D with the PDF $\mathcal{F}(\mathbf{x})$. Then

$$\frac{1}{D} N_q(\mathcal{X}) J_q(\mathcal{X}) \geq N_q(\mathcal{X}) [\det(\mathbb{J}_q(\mathcal{X}))]^{1/D} \geq 1, \quad (12)$$

where the Rényi parameter $q \geq 1$. A simple proof of the generalized isoperimetric inequality is provided in [30].

It is also worth noting that the relation (12) implies another important inequality. By using the fact that the Shannon entropy is maximized (among all PDF's with identical covariance matrix Σ) by the Gaussian distribution we have $N_1(\mathcal{X}) \leq \det(\Sigma)^{1/D}$ (see, e.g. [33]). If we further employ that \mathcal{I}_q is a monotonously decreasing function of q , see, e.g. [36, 34], we can write (recall that $q \geq 1$)

$$\frac{q^{1/(q-1)}}{e} N_q \leq N_1 = \frac{\exp(\frac{2}{D}\mathcal{I}_1)}{2\pi e} \leq \det(\Sigma)^{1/D}. \quad (13)$$

The isoperimetric inequality (12) then implies

$$\det(\Sigma(\mathcal{X})) \geq \frac{(q^{1/(q-1)})^D}{e^D \det(\mathbb{J}_q(\mathcal{X}))} \geq \frac{1}{e^D \det(\mathbb{J}_q(\mathcal{X}))}. \quad (14)$$

We can further use the inequality

$$\frac{1}{D} \text{Tr}(\mathbb{A}) \geq [\det(\mathbb{A})]^{1/D}, \quad (15)$$

(valid for any positive semi-definite $D \times D$ matrix \mathbb{A}) to write

$$\sigma^2(\mathcal{X}) = \frac{1}{D} \text{Tr}(\Sigma(\mathcal{X})) \geq \frac{Dq^{1/(q-1)}}{eJ_q(\mathcal{X})} \geq \frac{D}{eJ_q(\mathcal{X})}, \quad (16)$$

where σ^2 is an average variance per component.

Relations (14)-(16) represent the q -generalizations of the celebrated Cramér–Rao information inequality. In the limit of $q \rightarrow 1$ we recover the standard Cramér–Rao inequality which is widely used in statistical inference theory [37]. A final logical step needed to complete the proof of REPURs is represented by the so called generalized Stam inequality. To this end we first define the concept of *conjugate random variables*. We say that random vectors $\{\mathcal{X}_i\}$ and $\{\mathcal{Y}_i\}$ in \mathbb{R}^D are conjugate if their respective PDF's $\mathcal{F}(\mathbf{x})$ and $\mathcal{G}(\mathbf{y})$ can be written as

$$\mathcal{F}(\mathbf{x}) = |\psi_{\mathcal{F}}(\mathbf{x})|^2 / \|\psi_{\mathcal{F}}\|_2^2, \quad \mathcal{G}(\mathbf{y}) = |\psi_{\mathcal{G}}(\mathbf{y})|^2 / \|\psi_{\mathcal{G}}\|_2^2, \quad (17)$$

where the (generally complex) probability amplitudes $\psi_{\mathcal{F}}(\mathbf{x}) \in L_2(\mathbb{R}^D)$ and $\psi_{\mathcal{G}}(\mathbf{y}) \in L_2(\mathbb{R}^D)$ are mutual Fourier images, i.e., $\psi_{\mathcal{F}}(\mathbf{x}) = \hat{\psi}_{\mathcal{G}}(\mathbf{x})$ and $\psi_{\mathcal{G}}(\mathbf{y}) = \hat{\psi}_{\mathcal{F}}(\mathbf{y})$. With this we can state the generalized Stam inequality.

Generalized Stam inequality: Let $\{\mathcal{X}_i\}$ and $\{\mathcal{Y}_i\}$ be conjugate random vectors in \mathbb{R}^D . Then

$$16\pi^2 N_q(\mathcal{Y}) \geq [\det(\mathbb{J}_r(\mathcal{X}))]^{1/D}, \quad (18)$$

valid for any $r \in [1, \infty)$ and $q \in [1/2, 1]$ that are connected via relation $1/r + 1/q = 2$. A proof of the generalized Stam inequality is provided in [30].

Combining the isoperimetric inequality (12) together with the generalized Stam inequality (18) we obtain a one-parameter class of REP-based inequalities

$$N_{p/2}(\mathcal{X}) N_{q/2}(\mathcal{Y}) = N_{p/2}(\mathcal{X}) \frac{[\det(\mathbb{J}_{p/2}(\mathcal{X}))]^{1/D}}{[\det(\mathbb{J}_{p/2}(\mathcal{X}))]^{1/D}} N_{q/2}(\mathcal{Y}) \geq \frac{N_{q/2}(\mathcal{Y})}{[\det(\mathbb{J}_{p/2}(\mathcal{X}))]^{1/D}} \geq \frac{1}{16\pi^2}, \quad (19)$$

where p and q form now a Hölder double. By symmetry the role of q and p can be reversed. In Ref. [2] we presented an alternative (though more abstract) derivation of above REPURs which was based on the Beckner–Babenko theorem. There it was also proved that the inequality saturates if only if the distributions involved are Gaussian. Only exception to this rule is for the asymptotic values $p = 1$ and $q = \infty$ (or vice versa) where the saturation happens whenever the peak of $\mathcal{F}(\mathbf{x})$ and tail of $\mathcal{G}(\mathbf{y})$ (or vice versa) are Gaussian.

Importantly, since the REs are, in principle, measurable [38, 39, 40], the associated REPs are experimentally accessible. In addition, REPs of various orders are often used as convenient measures of entanglement — e.g., N_2 represents *tangle* τ (with $\sqrt{\tau}$ being *concurrence*) [41], $N_{1/2}$ is related to both *fidelity* F and *robustness* R of a pure state [42], N_∞ quantifies the Bures distance to the closest separable pure state [16], etc. For some recent applications of REs and REPs in quantum theory see, e.g., [43, 44, 45, 39].

3. Information distribution

Let $\mathcal{F}(\mathbf{x})$ be the PDF for the random variable \mathcal{X} . We define the *information random variable* $i_{\mathcal{X}}(\mathbf{x})$ so that $i_{\mathcal{X}}(\mathbf{x}) = \log_2 1/\mathcal{F}(\mathbf{x})$. In other words $i_{\mathcal{X}}(\mathbf{x})$ represents the information in \mathbf{x} with respect to $\mathcal{F}(\mathbf{x})$. In this connection it is expedient to introduce the cumulative distribution function for $i_{\mathcal{X}}(\mathcal{X})$ as

$$\wp(y) = \int_{-\infty}^y d\wp(i_{\mathcal{X}}) = \int_{\mathbb{R}^D} \mathcal{F}(\mathbf{x}) \theta(\log_2 \mathcal{F}(\mathbf{x}) + y) d\mathbf{x}. \quad (20)$$

The function $\wp(y)$ thus represents the probability that the random variable $i_{\mathcal{X}}(\mathcal{X})$ is less or equal than y . We have denoted the corresponding probability measure as $d\wp(i_{\mathcal{X}})$. Taking the Laplace transform of both sides of (20), we get

$$\mathcal{L}\{\wp\}(s) = \int_{\mathbb{R}^D} \mathcal{F}(\mathbf{x}) \frac{e^{s \log_2 \mathcal{F}(\mathbf{x})}}{s} d\mathbf{x} = \frac{\mathbb{E}[e^{s \log_2 \mathcal{F}}]}{s}, \quad (21)$$

where $\mathbb{E}[\dots]$ denotes the mean value with respect to \mathcal{F} . By assuming that $\wp(x)$ is smooth then the PDF associated with $i_{\mathcal{X}}(\mathcal{X})$ — the so-called *information PDF* — is

$$g(y) = \frac{d\wp(y)}{dy} = \mathcal{L}^{-1} \left\{ \mathbb{E} \left[e^{s \log_2 \mathcal{F}} \right] \right\} (y). \quad (22)$$

Setting $s = (p - 1) \log 2$ we have

$$\mathcal{L}\{g\}(s = (p - 1) \log 2) = \mathbb{E} \left[2^{(1-p)i_{\mathcal{X}}} \right]. \quad (23)$$

The mean here is taken with respect to the PDF g . Eq. (23) can also be written explicitly as

$$\int_{\mathbb{R}^D} d\mathbf{x} \mathcal{F}^p(\mathbf{x}) = \int_{\mathbb{R}} g(y) 2^{(1-p)y} dy. \quad (24)$$

Note that when \mathcal{F}^p is integrable for $p \in [1, 2]$ then (24) ensures that the moment-generating function for $g(x)$ PDF exists. So in particular, the moment-generating function exists when $\mathcal{F}(\mathbf{x})$ represents Lévy α -stable distributions, including the heavy-tailed stable distributions (i.e, PDFs with the Lévy stability parameter $\alpha \in (0, 2]$). The same holds for $\hat{\mathcal{F}}$ and $p' \in [2, \infty)$ due to the Beckner–Babenko theorem [2, 3, 46, 47].

4. Reconstruction theorem

Since $\mathcal{L}\{g\}(s)$ is the *moment-generating function* of the random variable $i_{\mathcal{X}}(\mathcal{X})$ one can generate all moments of the PDF $g(x)$ (if they exist) by taking the derivatives of $\mathcal{L}\{g\}$ with respect to s . From a conceptual standpoint, it is often more useful to work with cumulants rather than moments. Using the fact that the *cumulant generating function* is the logarithm of the moment-generating function, we see from (24) that the differential RE is a reparametrized version of the cumulant generating function of the random variable $i_{\mathcal{X}}(\mathcal{X})$. In fact, from (23) we have

$$\mathcal{I}_p(\mathcal{X}) = \frac{1}{(1-p)} \log_2 \mathbb{E} \left[2^{(1-p)i_{\mathcal{X}}} \right]. \quad (25)$$

To understand the meaning of REPURs we begin with the cumulant expansion (25), i.e.

$$p\mathcal{I}_{1-p}(\mathcal{X}) = \log_2 e \sum_{n=1}^{\infty} \frac{\kappa_n(\mathcal{X})}{n!} \left(\frac{p}{\log_2 e} \right)^n, \quad (26)$$

where $\kappa_n(\mathcal{X}) \equiv \kappa_n(i_{\mathcal{X}})$ denotes the n -th cumulant of the information random variable $i_{\mathcal{X}}(\mathcal{X})$ (in units of *bitsⁿ*). We note that

$$\kappa_1(\mathcal{X}) = \mathbb{E}[i_{\mathcal{X}}(\mathcal{X})] = \mathcal{H}(\mathcal{X}), \quad \kappa_2(\mathcal{X}) = \mathbb{E}[i_{\mathcal{X}}(\mathcal{X})^2] - (\mathbb{E}[i_{\mathcal{X}}(\mathcal{X})])^2, \quad (27)$$

i.e., they represent the Shannon entropy and *varentropy*, respectively. By employing the identity

$$\mathcal{I}_{1-p}(\mathcal{X}) = \frac{D}{2} \log_2 \left[2\pi(1-p)^{-1/p} N_{1-p}(\mathcal{X}) \right], \quad (28)$$

we can rewrite (26) in the form

$$\log_2 [N_{1-p}(\mathcal{X})] = \log_2 \left[\frac{(1-p)^{1/p}}{2\pi} \right] + \frac{2}{D} \sum_{n=1}^{\infty} \frac{\kappa_n(\mathcal{X})}{n!} \left(\frac{p}{\log_2 e} \right)^{n-1}. \quad (29)$$

From (29) one can see that

$$\kappa_n(\mathcal{X}) = \frac{nD}{2} (\log_2 e)^{n-1} \frac{d^{n-1} \log_2 [N_{1-p}(\mathcal{X})]}{dp^{n-1}} \Big|_{p=0} + \frac{D}{2} (\log_2 e)^n [(n-1)! + \delta_{1n} \log 2\pi], \quad (30)$$

which, in terms of the Grünwald–Letnikov derivative formula (GLDF) [48], allows us to write

$$\begin{aligned} \kappa_n(\mathcal{X}) &= \lim_{\Delta \rightarrow 0} \frac{nD}{2} \frac{(\log_2 e)^n}{\Delta^{n-1}} \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \log [N_{1+k\Delta}(\mathcal{X})] \\ &+ \frac{D}{2} (\log_2 e)^n [(n-1)! + \delta_{1n} \log 2\pi]. \end{aligned} \quad (31)$$

So, in order to determine the first m cumulants of $i_{\mathcal{X}}(\mathcal{X})$ we need to know all $N_1, N_{1+\Delta}, \dots, N_{1+(m-1)\Delta}$ entropy powers. In practice Δ corresponds to a characteristic resolution scale for the entropy index which will be chosen appropriately for the task at hand, but is typically of the order 10^{-2} . Note that the last term in (30) and (31) can be also written

$$\frac{D}{2} (\log_2 e)^n [(n-1)! + \delta_{1n} \log 2\pi] = \kappa_n(\mathcal{Z}_G^{\mathbb{I}}) \equiv \kappa_n(i_{\mathcal{Y}}), \quad (32)$$

with \mathcal{Y} being the random variable distributed with respect to the *Gaussian* distribution $\mathcal{Z}_G^{\mathbb{I}}$ with the *unit* covariance matrix.

When all the cumulants exist then the problem of recovering the underlying PDF for $i_{\mathcal{X}}(\mathcal{X})$ is equivalent to the *Stieltjes* moment problem [49]. Using this connection, there are a number of ways to proceed; the PDF in question can be reconstructed e.g., in terms of sums involving orthogonal polynomials (e.g., the Gram–Charlier A series or the Edgeworth series [50]), the inverse Mellin transform [51] or via various maximum entropy techniques [52]. Pertaining to this, the theorem of Marcinkiewicz [53] implies that there are no PDFs for which $\kappa_m = \kappa_{m+1} = \dots = 0$ for $m \geq 3$. In other words, the cumulant generating function cannot be a finite-order polynomial of degree greater than 2. The only exceptions to Marcinkiewicz’s theorem, are the *Gaussian* PDFs which can have the first two cumulants nontrivial and $\kappa_3 = \kappa_4 = \dots = 0$. Thus, apart from the special case of Gaussian PDFs where only N_1 and $N_{1+\Delta}$ are required, one needs to work with as many entropy powers $N_{1+k\Delta}$, $k \in \mathbb{N}$ as possible to receive as much information as possible about the structure of the underlying PDF. In theory, the whole infinite tower of REPURs would be required to uniquely specify a system’s information PDF. Note, that for *Gaussian* information PDFs one needs only N_1 and $N_{1+\Delta}$ to reconstruct the PDF uniquely. From (29) and (31) we see that knowledge of N_1 corresponds to $\kappa_1(\mathcal{X}) = \mathcal{H}(\mathcal{X})$ while $N_{1+\Delta}$ further determines κ_2 , i.e. the varentropy. Since N_1 is involved [via (31)] in the determination of all cumulants, it is the most important entropy power in the tower.

We should stress, that the focus of the reconstruction theorem we present is on cumulants κ_n which can be directly used for a shape estimation of $g(x)$ but not $\mathcal{F}(\mathbf{x})$. However, by knowing $g(y)$ we have a complete “information scan” of $\mathcal{F}(\mathbf{x})$. Such an information scan is, however, not unique, indeed two PDFs that are rearrangements of each other – i.e., *equimeasurable* PDFs, have identical $\wp(y)$ and $g(y)$. Even though equimeasurable PDFs cannot be distinguished via their entropy powers, they can be, as a rule, distinguished via their respective momentum-space PDFs and associated entropy powers. So the information scan has a tomographic flavor to it. From the multi-peak structure of $g(y)$ one can determine the *number* and *height* of the stationary points. These are invariant characteristics of a given family of equimeasurable PDFs.

5. Information scan of the state PDF

With knowledge of the entropy powers, the question now is how we can reconstruct the information distribution $g(x)$. The inner workings of this will be now explicitly illustrated with the (generalized) Gram-Charlier A expansion. However, other – often more efficient methods – are also available [50]. Let κ_n be cumulants obtained from entropy powers and let $G(x)$ be some reference PDF whose cumulants are γ_k . The information PDF $g(x)$ can be then written as [50]

$$g(x) = \exp \left[\sum_{k=1}^{\infty} (\kappa_k - \gamma_k) (-1)^k \frac{(d^k/dx^k)}{k!} \right] G(x). \quad (33)$$

With the hindsight we choose the reference PDF $G(x)$ to be a shifted gamma PDF, i.e.

$$G(x) \equiv \mathcal{G}(x|a, \alpha, \beta) = \frac{e^{-(x-a)/\beta} (x-a)^{\alpha-1}}{\beta^{\alpha} \Gamma[\alpha]}, \quad (34)$$

with $a < x < \infty$, $\beta > 0$, $\alpha > 0$. In doing so, we have implicitly assumed that the $\mathcal{F}(y)$ PDF is in the first approximation equimeasurable with the Gaussian PDF. To reach a corresponding matching we should choose $a = \log_2(2\pi\sigma^2)/2$, $\alpha = 1/2$ and $\beta = \log_2 e$. Using the fact that [54]

$$\beta^{k+1/2} \frac{d^k \mathcal{G}(x|a, 1/2, \beta)}{k! dx^k} = \left(\frac{x-a}{\beta} \right)^{-k} L_k^{(-1/2-k)} \left(\frac{x-a}{\beta} \right) \mathcal{G}(x|a, 1/2, \beta), \quad (35)$$

(L_k^δ is an associated Laguerre polynomial of order k with parameter δ). Given that $\kappa_1 = \gamma_1 = \alpha\beta + a = \log_2(2\pi\sigma^2 e)/2$, and $\gamma_k = \Gamma(k)\alpha\beta^k = (\log_2 e)^k/2$ for $k > 1$ we can write (33) as

$$g(x) = \mathcal{G}(x|a, 1/2, \beta) \left[1 + \frac{(\kappa_2 - \gamma_2)}{\beta^{1/2}(x-a)^2} L_2^{(-5/2)} \left(\frac{x-a}{\beta} \right) - \frac{(\kappa_3 - \gamma_3)}{\beta^{1/2}(x-a)^3} L_3^{(-7/2)} \left(\frac{x-a}{\beta} \right) + \dots \right]. \quad (36)$$

For the Gram–Charlier A expansion various formal convergence criteria exist (see, e.g., [50]). In particular, the expansion for nearly Gaussian equimeasurable PDFs $\mathcal{F}(y)$ converges quite rapidly and the series can be truncated fairly quickly. Since in this case one needs fewer κ_k 's in order to determine $g(x)$, only EPs in the small neighborhood of the index 1 are needed. On the other hand, the further the $\mathcal{F}(y)$ is from Gaussian the higher orders of κ_k are required to determine $g(x)$, and hence a wider neighborhood of the index 1 will be needed for EPs.

6. (Un)Balanced Cat State and reconstruction theorem

We now demonstrate an example of the reconstruction in the context of QM. Specifically, we consider cat states that are often considered in the foundations of quantum physics as well as in various applications, including solid state physics [55] and quantum metrology [56]. The form of the state we consider is $|\psi\rangle = \mathcal{N}(|0\rangle + \nu|\alpha/\nu\rangle)$ where, $\mathcal{N} = [1 + 2\nu \exp(-\alpha^2/2\nu^2) + \nu^2]^{-1/2}$ is the normalization factor, $|0\rangle$ is the vacuum state, $\nu \in \mathbb{R}$ a weighting factor and $|\alpha\rangle$ is the coherent state given by ($\alpha \in \mathbb{R}$)

$$|\alpha\rangle = e^{-\alpha^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle, \quad (37)$$

For $\nu = 1$ we refer to the state as a *balanced cat state* (BCS) and for $\nu \neq 1$, as an *unbalanced cat state* (UCS). Changing the basis of $|\psi\rangle$ to the eigenstates of the general quadrature operator

$$\hat{Y}_\theta = \frac{1}{\sqrt{2}} \left(\hat{a} e^{-i\theta} + \hat{a}^\dagger e^{i\theta} \right), \quad (38)$$

where, \hat{a} and \hat{a}^\dagger are the creation and annihilation operators of the electromagnetic field, we find the PDF for the general quadrature variable y_θ to be

$$\mathcal{F}(y_\theta) = \mathcal{N}^2 \pi^{-\frac{1}{2}} e^{-y_\theta^2} \left| 1 + \nu \exp \left[-\frac{\alpha^2}{\nu^2 2} \left(1 + e^{2i\theta} - 2\sqrt{2} e^{i\theta} \frac{\nu}{\alpha} y_\theta \right) \right] \right|^2, \quad (39)$$

Setting $\theta = 0$ and $\nu = 1$ returns the PDF of the BCS for the position-like variable y_0 . With this, the Rényi EPs $N_{1-p}(\chi)$ are calculated and found to be constant across varying p . This is because $\mathcal{F}(y_0)$ for the BCS is in fact a piecewise rearrangement of a Gaussian PDF (yet has an overall non-Gaussian structure), thus $N_{1-p}(\chi) = \sigma^2$ for all p , where σ^2 is the variance of the ‘would be Gaussian’. Taking the reference PDF to be $G(x) = \mathcal{G}(x|a, \alpha, \beta)$, with $a = \log_2(2\pi\sigma^2)/2$, $\alpha = 1/2$ and $\beta = \log_2(e)$, it is evident that $(\kappa_k - \gamma_k) = 0$ for all $k \geq 1$, and from the Gram–Charlier A series (33), a perfect matching in the reconstruction is achieved. Furthermore, it can be shown that the variance of (39) increases with α , i.e. the variance increases as the peaks of the PDF diverge, which is in stark contrast to the Rényi EPs which remain constant for increasing α . This reveals the shortcomings of variance as a measure of uncertainty for non-Gaussian PDFs.

The peaks, located at $\mathcal{F}(y_\theta) = 2^{-a_j^+}$, where j is an index labelling the distinct peaks, gives rise to sharp singularities in the target $g(x)$. With regard to the BCS position PDF, distributions of

the conjugate parameter $\mathcal{F}(y_{\pi/2})$ distinguish $\mathcal{F}(y_0)$ from it's equimeasurable Gaussian PDF and hence the Rényi EPs also distinguish the different cases. The number of available cumulants k is computationally limited but as this grows, information about the singularities will be recovered in the reconstruction. In the following, we show how the tail convergence and location of a singularity for $g(x)$ can be reconstructed using $k = 5$. We consider the case of a UCS with $\nu = 0.97$, $\alpha = 10$ and we take $\theta = 0$ in equation (39) to find the PDF in the y_0 quadrature which is non-Gaussian for all piecewise rearrangements. As such, all REPs N_{1-p} vary with p and consequently all cumulants κ_k carry information on $g(x)$. Here we choose to reconstruct the UCS information distribution by means of the Edgeworth series [50] so that

$$g(x) = \exp \left[n \sum_{j=2}^{\infty} (\kappa_j - \gamma_j) \frac{(-1)^j}{j!} \frac{d^j}{dx^j} n^{-j/2} \right] G(x), \quad (40)$$

where the reference PDF $G(x)$ is again the shifted gamma distribution. Using the Edgeworth series, the information PDF is approximated by expanding in orders of n , which has the advantage over the Gram–Charlier A expansion discussed above of bounding the errors of the approximation. For the particular UCS of interest, expanding to order $n^{-3/2}$ reveals convergence toward the analytic form of the information PDF shown as the target line in Fig. 1. This shows that, for a given characteristic resolution, control over the first five Rényi EPs can be enough for a useful information scan of a quantum state with an underlying non-Gaussian PDF. In the example shown in Fig. 1 we see that the information scan accurately predicts the tail behavior as well as the location of the singularity, which corresponds to the second (lower) peak of $\mathcal{F}(y_0)$.

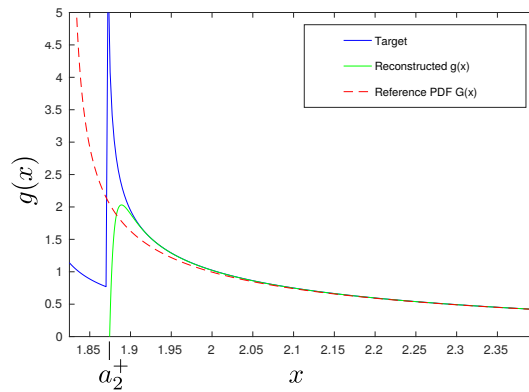


Figure 1. Reconstructed information distribution of an unbalanced cat state with $\nu = 0.97$ and $\alpha = 10$. The Edgeworth expansion has been used here to order $n^{-3/2}$ requiring control of the first five REPs. Good convergence of the tail behaviour is evident as well as the location of the singularity corresponding to the second peak; a_2^+ corresponds to the value of x at the point of intersection with the second (lower) peak of $\mathcal{F}(y_0)$.

7. Conclusions

In the first part of this paper we have presented a new proof of a one-parameter class of Rényi-entropy-power-based URs for pairs of observables in an infinite-dimensional Hilbert space. This was done with the help of the generalized isoperimetric and Stam inequality. This proof substantiates our earlier version in [2, 3] which was based on the Babenko–Beckner inequality.

The mathematical language employed in the new proof (i.e., Fisher information matrix, Cramér–Rao inequality, etc.) is much closer to the language familiar in quantum information theory and, in particular, quantum metrology, and so makes stronger connections to these fields.

In the second part we present the method for reconstructing the underlying information PDF associated with a given quantum state from known entropy powers. The details of this reconstruction procedure have been explained in terms of what we call an information scan and mathematically formalized in a reconstruction theorem. A numerical implementation of the reconstruction theorem was discussed and we demonstrated its utility in a proof-of-principle analysis by reconstructing the information PDF of a quantum state described by a non-Gaussian probability distribution. In this case it was found that control of the first five REPs gave enough information for a meaningful reconstruction of the information PDF. In particular we demonstrated the superiority of REPs over variance as a measure of uncertainty for the important class of non-Gaussian states — (un)balanced cat states.

REPURs, together with the reconstruction theorem that has been established and discussed here, can be used for various problems in the theory of quantum information and metrology. Specific examples include separability conditions and the characterization of multipartite entanglement or improved witnesses of quantum entanglement and (weak) measurements in the presence of quantum memory. These avenues are currently being actively pursued.

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